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Shallow-water flows past slender bodies

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The problem solved concerns the disturbance to a stream of shallow water due to an immersed slender body, with special application to the steady motion of ships in shallow water. Formulae valid to first order in slenderness are given for the wave resistance and vertical forces at both sub- and supercritical speeds. The vertical forces are used to predict sinkage and trim of ships and satisfactory comparisons with model experiments are made.

1. Introduction

This paper contains a systematic investigation of the problem of shallow-water flow past a fixed slender obstacle in a stream. This problem has a particular bearing on, and was suggested by, the behaviour of ships moving in still water of restricted depth, but may have application to a variety of problems involving shallow water, such as river flows past obstacles. However, for definiteness we shall here refer to the slender obstacle as a ship, and its surface as the hull.

The case of a thin obstacle which extends with vertical sides all the way from the free surface to the bottom is well known, and in fact under the assumptions of linearized shallow-water theory this problem is entirely congruent to the steady aerodynamics of a thin wing. The first solution appears to have been given by Michell in his famous wave-resistance paper (Michell 1898); the result was also obtained by Joukowski in 1903 (Kostyukov 1959) and more recently by Laitone (Wehausen & Laitone 1960). Michell found that at subcritical speeds ($F_h = U/\sqrt{(gh)} < 1$, where U is the stream speed, g the acceleration of gravity and h the stream depth) the drag force on the obstacle is zero, while at supercritical speeds the drag is

$$R = \frac{\rho U^2 h}{2\sqrt{(F_h^2-1)}}\int dx[B'(x)]^2, \tag{1.1}$$

where $B(x)$ is the thickness distribution of the strut. This result is physically quite reasonable by analogy with aerodynamics, since the Froude number F_h plays the same role as the Mach number, but a prediction of zero for subcritical wave resistance did not seem reasonable to naval architects, and this part of Michell's paper has been largely ignored. In any case he treated a situation where the obstacle was not particularly ship-like, and it was not obvious how to extend the work to the case where the 'ship' does not have vertical sides nor meets the bottom.

The geometry of Michell's strut is such that the vertical force and moment are zero. In practice it is found that there is always a net downward force on a ship moving in shallow water, and often also a trimming moment. The phenomenon is known as 'squat', and is a matter of some concern to every pilot and ship's master who is responsible for taking a large vessel into harbours of restricted depth. At conventional speeds of conventional ships the downward force is capable of producing a 'sinkage' of one or two feet, which must be taken into account under some circumstances; the effect is even more pronounced for large high-speed vessels such as aircraft carriers where sinkages of 6–8 ft. can occur. In order to produce a theory for these vertical forces and moments we must abandon the idea of a strut extending from surface to bottom of the water, and attempt to take more account of the actual geometry of the body.

The analysis which follows assumes the ship to be slender in the sense that it is longer than it is broad or deep, and uses the now well-known technique of matched asymptotic expansions (or 'inner and outer expansions') to construct an approximate solution. This technique, developed by Kaplun for use in boundary-layer theory, has in recent years been used with success to solve a number of difficult singular-perturbation problems, and with the publication of a textbook (Van Dyke 1964) devoted to it, may be said to have become firmly established as a basic working tool of applied mathematics. The results obtained here by this means include an extension of Michell's wave-resistance formulae to more general hull geometries, together with expressions for the vertical force and moment at both sub- and supercritical speeds. The latter are used to give the sinkage and trim displacements of a ship, and satisfactory comparisons with experiments are shown.

2. Exact formulation of the problem

Suppose a fixed slender obstacle (ship) is at or near the free surface of a stream U of shallow water. The stream flows from left to right in the co-ordinate system of figure 1, and is of depth h. The fluid is taken to be inviscid and incompressible and the flow steady and irrotational, so that there exists a disturbance potential ϕ satisfying Laplace's equation and tending to zero suitably at infinity, such that the total fluid velocity is $U\nabla(x+\phi)$.

On the ship's hull surface the normal velocity vanishes, i.e.

$$\partial(x+\phi)/\partial n = 0. \tag{2.1}$$

The form of the normal derivative $\partial/\partial n$ may be written out explicitly in terms of any given equation for the hull surface. For instance, if we are given

$$y = Y(x,z) \tag{2.2}$$

as the equation of the hull, then equation (2.1) becomes

$$\frac{\partial\phi}{\partial y} - \left(1+\frac{\partial\phi}{\partial x}\right)\frac{\partial Y}{\partial x} - \frac{\partial\phi}{\partial z}\frac{\partial Y}{\partial z} = 0. \tag{2.3}$$

On the bottom, assumed to be a plane surface $z = -h$, we have

$$\partial\phi/\partial z = 0, \tag{2.4}$$

while the boundary conditions on the unknown free surface

$$z = \zeta(x,y) \tag{2.5}$$

are firstly that the pressure vanishes, i.e.

$$-\frac{2g\zeta}{U^2} = 2\phi_x + \phi_x^2 + \phi_y^2 + \phi_z^2, \tag{2.6}$$

and secondly that the free surface is a streamline, i.e.

$$\phi_z = \zeta_x + \phi_x \zeta_x + \phi_y \zeta_y, \tag{2.7}$$

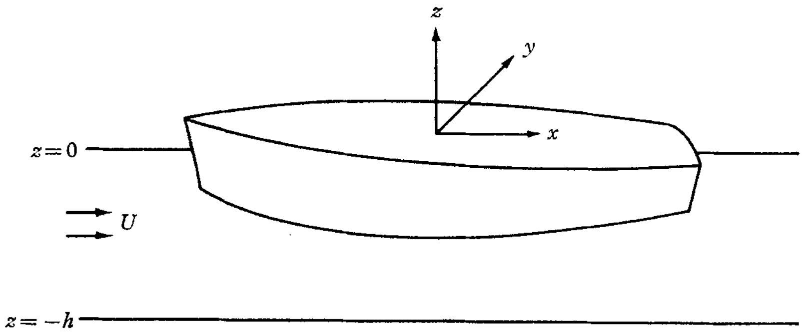

FIGURE 1. Co-ordinate system.

The sequel consists of an approximate solution to the above boundary-value problem for Laplace's equation, based on the assumptions that the ship is slender and the water shallow. Specifically, if B is the beam, T the draft, L the length of the ship, we assume

$$h/L,\ B/L,\ T/L = O(\epsilon) \tag{2.8}$$

and

$$F_h^2 = U^2/gh = O(1), \tag{2.9}$$

where ϵ is a small parameter. Incidentally, the last condition (2.9) above implies that the conventional Froude number $F_L = U/\sqrt{(gL)}$ is small, of order $\epsilon^{\frac{1}{2}}$. This condition ensures that we are in a range where wave-making is significant; it has no other purpose, and indeed a re-examination of all approximations made in this paper indicates that the results are still valid in the limit as the free surface becomes a rigid wall, i.e. $g \to \infty$ or $F_h \to 0$. Thus we are at the same time solving the problem of streaming past a slender obstacle sandwiched between two close parallel plates.

3. The outer expansion

We now propose to solve the problem systematically by putting together two asymptotic expansions, one called the outer expansion and valid far from the ship, and one the inner expansion valid near to the ship. In this section we pursue the outer expansion in an 'outer region' defined by the following orders of magnitude of the co-ordinates

$$x, y = O(1), \quad z = O(\epsilon) \tag{3.1}$$

(since these co-ordinates are dimensional quantities, the order statements are to be interpreted as describing their magnitude with respect to the ship length L; this convention will be followed throughout). It is now assumed that in this region ϕ possesses an asymptotic expansion with respect to ϵ of the form

$$\phi = \phi^{(1)} + \phi^{(2)} + \phi^{(3)} + \ldots, \tag{3.2}$$

which is ordered with respect to ϵ so that $\phi^{(n+1)} = o(\phi^{(n)})$ if and only if x, y, z are of the orders of magnitude given in equations (3.1). It is not in general necessary to further restrict the nature of the expansion at this time; however, hindsight tells us that in this problem the series at least starts out like a power series. We shall therefore take $\phi^{(n)} = O(\epsilon^n)$ in order to simplify the analysis; if it should happen that this assumption leads to inconsistencies at a later stage we should then return to a more general expansion, which might, for instance, involve terms of order $\epsilon^n \log \epsilon$ (see Van Dyke 1964, p. 200, for further discussion of this point).

Now by collecting terms of like order of magnitude in the Laplace equation

$$\phi_{zz} = -\phi_{xx} - \phi_{yy} = -\nabla^2_{(x,y)}\phi, \tag{3.3}$$

we have successively

$$\left.\begin{aligned} &\phi^{(1)}_{zz} = 0, \quad \phi^{(2)}_{zz} = 0, \\ &\phi^{(3)}_{zz} = -\nabla^2_{(x,y)}\phi^{(1)}, \quad \phi^{(4)}_{zz} = -\nabla^2_{(x,y)}\phi^{(2)}, \end{aligned}\right\} \tag{3.4}$$

etc., where we are using the notation $\nabla^2_{(x,y)}$ to indicate a two-dimensional Laplacian in the (x, y)-plane. Thus $\phi^{(1)}$, $\phi^{(2)}$ are linear in z, $\phi^{(3)}$, $\phi^{(4)}$ are quadratic in z, etc. We can immediately solve the above equations, making use of the bottom condition $\phi^{(n)}_z = 0$ on $z = -h$ to obtain:

$$\left.\begin{aligned} &\phi^{(1)} = \Psi^{(1)}(x,y), \quad \phi^{(2)} = \Psi^{(2)}(x,y), \\ &\phi^{(3)} = \Psi^{(3)}(x,y) - \tfrac{1}{2}(z+h)^2\,\nabla^2_{(x,y)}\Psi^{(1)}(x,y), \end{aligned}\right\} \tag{3.5}$$

etc., where $\Psi^{(n)}(x,y) = O(\epsilon^n)$ $(n = 1, 2, \ldots)$, is a set of (so far arbitrary) functions of x and y.

The partial differential equations satisfied by the unknown functions $\Psi^{(n)}$ are found by substituting the outer expansion in the free-surface conditions (2.6), (2.7). Thus, from the pressure condition we have

$$-(2g/U^2)\zeta = 2\Psi^{(1)}_x + 2\Psi^{(2)}_x + 2\Psi^{(3)}_x + (\Psi^{(1)}_x)^2 + (\Psi^{(1)}_y)^2 + O(\epsilon^4).$$

Notice that ζ is $O(\epsilon^2)$ since $U^2/g = O(\epsilon)$ with respect to L by (2.9). That is, ζ itself has an asymptotic expansion

$$\zeta = \zeta^{(2)} + \zeta^{(3)} + \zeta^{(4)} + \ldots, \tag{3.6}$$

where

$$\begin{aligned} &\zeta^{(2)} = -(U^2/g)\,\Psi^{(1)}_x, \\ &\zeta^{(3)} = -(U^2/g)\,[\Psi^{(2)}_x + \tfrac{1}{2}(\Psi^{(1)}_x)^2 + \tfrac{1}{2}(\Psi^{(1)}_y)^2], \quad \text{etc.} \end{aligned} \tag{3.7}$$

Now the kinematic free-surface condition gives

$$\begin{aligned} &(-h + \zeta^{(2)})\,\nabla^2_{(x,y)}\Psi^{(1)} - h\nabla^2_{(x,y)}\Psi^{(2)} + O(\epsilon^4) \\ &\qquad = \zeta^{(2)}_x + \zeta^{(3)}_x + \zeta^{(2)}_x\Psi^{(1)}_x + \zeta^{(2)}_y\Psi^{(1)}_y + O(\epsilon^4), \end{aligned} \tag{3.8}$$

i.e.

$$\begin{aligned} &\nabla^2_{(x,y)}\Psi^{(1)} = -\zeta^{(2)}_x/h, \\ &\nabla^2_{(x,y)}\Psi^{(2)} = -1/h\,\zeta^{(2)}\nabla^2_{(x,y)}\Psi^{(1)} - 1/h\,(\zeta^{(3)}_x + \zeta^{(2)}_x\Psi^{(1)}_x + \zeta^{(2)}_y\Psi^{(1)}_y). \end{aligned}$$

Finally, on substitution of the previously determined expressions for $\zeta^{(2)}, \zeta^{(3)}, \ldots,$ we have as the equations for the $\Psi^{(n)}$,

$$\left(1 - \frac{U^2}{gh}\right)\Psi^{(1)}_{xx} + \Psi^{(1)}_{yy} = 0, \tag{3.9}$$

$$\left(1 - \frac{U^2}{gh}\right)\Psi^{(2)}_{xx} + \Psi^{(2)}_{yy} = \frac{U^2}{gh}\left[\left(2 + \frac{U^2}{gh}\right)\Psi^{(1)}_x\Psi^{(1)}_{xx} + 2\Psi^{(1)}_y\Psi^{(1)}_{xy}\right], \tag{3.10}$$

etc. Equation (3.9) is the usual equation of linearized shallow-water theory, which we should have expected in this problem; indeed the foregoing is merely a special derivation of this well-known theory. Of course the equation is also identical with that for linearized aerodynamics in two dimensions, if we identify $F_h = U/\surd(gh)$ with the free-stream Mach number. Equation (3.10) is an inhomogeneous version of the same equation with a right-hand side involving a combination of derivatives of the previously determined $\Psi^{(1)}$; the equations satisfied by further terms $\Psi^{(3)}$, $\Psi^{(4)}$, etc., will clearly be of a similar nature but with even more complicated right-hand sides. Since we shall not in this paper use even $\Psi^{(2)}$ in the above form, there is little point in writing down further terms, although there are no conceptual reasons why this could not be done if required.

To an observer in the outer region the ship appears to have collapsed onto the plane $y = 0$ as $\epsilon \to 0$, since for this observer y must be $O(1)$ but the beam of the ship is of $O(\epsilon)$. On the other hand, the z co-ordinates of both the ship and the observer are $O(\epsilon)$, so that the draught of the ship remains (relatively) finite according to this observer as $\epsilon \to 0$. Thus we must seek solutions of equations (3.9) and (3.10) which are analytic everywhere except possibly on the plane $y = 0$, and this can be done by methods familiar in aerodynamics.

For instance, if $F_h > 1$, equation (3.9) is formally congruent to the one-dimensional wave equation, and the general solution which is symmetric in y and analytic for all $y \neq 0$ is

$$\Psi^{(1)}(x,y) = \Psi(x - \surd(F_h^2-1)\,|y|),$$

for some arbitrary function

$$\Psi(x) = \Psi^{(1)}(x,0).$$

We prefer to take

$$-\surd(F_h^2-1)\,d\Psi/dx = \Psi_y^{(1)}(x,0_+)$$

as our arbitrary function, setting

$$\Psi^{(1)}(x,y) = -\frac{1}{\surd(F_h^2-1)}\int_{-\infty}^{[x-\surd(F_h^2-1)|y|]} d\xi\,\Psi_y^{(1)}(\xi,0_+). \tag{3.11}$$

On the other hand, if $F_h < 1$, the equation (3.9) is elliptic, and its general solution may be written down using Green's theorem in the form

$$\Psi^{(1)}(x,y) = \int_{-\infty}^{\infty} d\xi\,\Psi_y^{(1)}(\xi,0_+)\,G(x-\xi,y), \tag{3.12}$$

with

$$G(x,y) = \frac{1}{2\pi\surd(1-F_h^2)}\log\surd\{x^2+(1-F_h^2)y^2\}$$

as the Green's function (or unit source potential). The supercritical result (3.11) may also be written in the form (3.12) if we set

$$G(x,y) = -\frac{1}{\surd(F_h^2-1)}H(x-\surd(F_h^2-1)\,|y|),$$

where H is the Heaviside unit step function. Near $y = 0_\pm$, the general solution (3.12) has a series expansion of the form

$$\Psi^{(1)}(x,y) = \Psi^{(1)}(x,0) + |y|\,\Psi_y^{(1)}(x,0_+) - \tfrac{1}{2}y^2(1-F_h^2)\,\Psi_{xx}^{(1)}(x,0) + \ldots, \tag{3.13}$$

where

$$\Psi'^{(1)}(x,0) = \begin{Bmatrix} -\dfrac{1}{\pi\sqrt{(1-F_h^2)}}\displaystyle\int_{-\infty}^{\infty} d\xi \Psi'^{(1)}_y(\xi, 0_+) \log|x-\xi| & (F_h < 1) \\ -\dfrac{1}{\sqrt{(F_h^2-1)}}\displaystyle\int_{-\infty}^{x} d\xi \Psi'^{(1)}_y(\xi, 0_+) & (F_h > 1) \end{Bmatrix}. \quad (3.14)$$

The solution for $\Psi'^{(2)}$ and indeed all higher approximations may also be written down by means of Green's theorem. For if we denote the right-hand side of equation (3.10) by $\Omega^{(2)}(x, y)$, then the general solution of (3.10) in $y \geq 0$ is

$$\Psi'^{(2)}(x,y) = \int_{-\infty}^{\infty} d\xi \Psi'^{(2)}_y(\xi, 0_+)\, G(x-\xi, y) + \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, \Omega^{(2)}(\xi,\eta)\, G(x-\xi, y-\eta). \quad (3.15)$$

That is, the boundary source distribution of (3.11) must be augmented by a spatial source distribution of density $\Omega^{(2)}$. Clearly the solution for all $\Psi'^{(n)}$ will be of this form with $\Omega^{(n)}$ as the right-hand side of the general equation for $\Psi'^{(n)}$. In practice, of course, the solution (3.15) is already extremely complicated for $\Psi'^{(2)}$, since $\Omega^{(2)}$ involves $\Psi'^{(1)}$ quadratically so that the double integral is in reality a quadruple integral in the original unknown function $\Psi'^{(1)}_y(x, 0_+)$. We may note in passing, however, that in the limit as $F_h \to 0$, equation (3.10) indicates that $\Omega^{(2)} \to 0$, so that the solution for $\Psi'^{(2)}$ is of the same form as that for $\Psi'^{(1)}$.

In order to find the arbitrary functions $\Psi'^{(1)}_y(x, 0_+)$, $\Psi'^{(2)}_y(x, 0_+))$, etc., we must match the inner and outer expansions, so we now proceed to determine a number of terms of the inner expansion.

4. The inner expansion

The inner region is defined by the following orders of magnitude of the co-ordinates:

$$x = O(1), \quad y, z = O(\epsilon). \quad (4.1)$$

Thus, as seen by an observer in this region, neither breadth nor depth of the ship is small. We take an asymptotic expansion of the form

$$\phi = \Phi^{(1)} + \Phi^{(2)} + \ldots, \quad (4.2)$$

which is ordered with respect to ϵ only so long as x, y, z are of the magnitudes above. As with the outer expansion, we can assume, until and unless we have evidence to the contrary, that the series is of the nature of a power series in ϵ, i.e. that $\Phi^{(n)} = O(\epsilon^n)$.

Now Laplace's equation is

$$\phi_{yy} + \phi_{zz} = \nabla^2_{(y,z)}\phi = -\phi_{xx}, \quad (4.3)$$

so that, on collecting terms of like order in ϵ, we have

$$\left.\begin{aligned} \nabla^2_{(y,z)}\Phi^{(1)} &= 0, & \nabla^2_{(y,z)}\Phi^{(2)} &= 0, \\ \nabla^2_{(y,z)}\Phi^{(3)} &= -\Phi^{(1)}_{xx}, & \nabla^2_{(y,z)}\Phi^{(4)} &= -\Phi^{(2)}_{xx}, \end{aligned}\right\} \quad (4.4)$$

etc. That is, the individual terms in the asymptotic expansion satisfy two-dimensional Laplace or Poisson equations in the (y, z)-plane.

The limiting boundary conditions on the hull surface and on the free surface are the same as for a slender body in water of infinite depth (Tuck 1964), and only the results will be quoted here. On the hull surface we have

$$\left.\begin{aligned} \partial\Phi^{(1)}/\partial N &= 0, & \partial\Phi^{(2)}/\partial N &= v_N, \\ \partial\Phi^{(3)}/\partial N &= v_N \Phi^{(1)}_x, & \partial\Phi^{(4)}/\partial N &= v_N \Phi^{(2)}_x, \end{aligned}\right\} \tag{4.5}$$

etc., where N is a normal to the cross-section curve at x, and v_N is a definite function of the hull shape. Specifically, if we use the defining equation (2.2), we have

$$v_N = \frac{\partial Y}{\partial z} \bigg/ \sqrt{\left\{1 + \left(\frac{\partial Y}{\partial z}\right)^2\right\}} \tag{4.6}$$

(this is a slender-body approximation to the x-component of the unit normal to the hull). It follows that $v_N = O(\epsilon)$ and that (Tuck 1964)

$$\int v_N \, dl = S'(x), \tag{4.7}$$

where the integral is taken round the immersed cross-section of the hull at x and where $S(x)$ is the area of that section below the plane $z = 0$. Thus the flux of $\Phi^{(2)}$ through the hull section at x is $S'(x)$. On the free surface we have successively (Tuck 1964)

$$\left.\begin{aligned} &\partial\Phi^{(1)}/\partial z = 0, \quad \partial\Phi^{(2)}/\partial z = 0, \\ &\partial\Phi^{(3)}/\partial z = -(U^2/g)\,[\Phi^{(1)}_{xx} + 2\Phi^{(1)}_y \Phi^{(1)}_{xy} + \Phi^{(1)}_x \Phi^{(1)}_{yy} + \tfrac{3}{2}\Phi^{(1)2}_y \Phi^{(1)}_{yy}], \end{aligned}\right\} \tag{4.8}$$

etc. Finally, on the bottom $\qquad \partial\Phi^{(n)}/\partial z = 0 \qquad$ (4.9)
for all $n = 1, 2, \ldots$.

We now have a series of classical two-dimensional Neumann problems to solve separately for each cross-section x. Each Neumann problem will be indeterminate to the extent of a constant in its plane; that is, to the extent of an arbitrary function of x only. For instance, $\Phi^{(1)}$ is clearly at most a constant with respect to y and z, since it has zero normal derivative on all boundaries in the (y, z)-plane; i.e.

$$\Phi^{(1)} = f_1(x) \tag{4.10}$$

for some arbitrary function $f_1(x)$ which must be found by matching with the outer expansion.

In order to solve for $\Phi^{(2)}$, let us split off the indeterminate part, writing

$$\Phi^{(2)} = f_2(x) + \Phi^{(2)}_*, \tag{4.11}$$

where $f_2(x)$ is arbitrary but $\Phi^{(2)}_*$ is uniquely defined by applying a suitable boundary condition at infinity. For instance, we can require

$$\Phi^{(2)}_* \to u\,|y| + o(1) \quad \text{as} \quad y \to \pm\infty, \tag{4.12}$$

where the constant $u = u(x)$ is determinable from conservation of mass, and where the '$o(1)$' term implies that the difference between $\Phi^{(2)}_*$ and the stream $u\,|y|$ tends to zero† as $y \to \pm\infty$. In fact it is clear that

$$u = (1/2h)\, S'(x) \tag{4.13}$$

† By use of Green's theorem it can be shown that this difference is actually of order $e^{-|y|/h}$.

since one half of the flux $S'(x)$ across the hull section is channelled each way into width h as $y \to \pm\infty$. Thus $\Phi_*^{(2)}$ is a uniquely determined potential which may be found by any of the classical methods for solving two-dimensional Neumann problems, analytically or numerically.

The third term $\Phi^{(3)}$ in the inner expansion may be treated similarly. Some extra care is needed, however, since $\Phi^{(3)}$ satisfies a Poisson rather than Laplace equation, and further involves a non-zero normal velocity across the free surface; we quote only the form of $\Phi^{(3)}$ resulting from splitting off contributions from these additional effects, namely,

$$\Phi^{(3)} = f_1'\,\Phi_*^{(2)} - \tfrac{1}{2}f_1''[(1-F_h^2)y^2 + (z+h)^2F_h^2] + f_3, \tag{4.14}$$

where $f_1(x)$ is the arbitrary 'constant' associated with $\Phi^{(1)}$, $f_3(x)$ is a new arbitrary 'constant', and $\Phi_*^{(2)}$ is as defined previously.

5. Matching

The analytical process of matching may be defined in a number of ways, ranging from crude but useful, to rigorous but indigestible; Chapter V of Van Dyke's book discusses this question in detail. The usual compromise is to use his equation (5.24), namely:

'The m-term inner expansion of the (n-term outer expansion) = the n-term outer expansion of the (m-term inner expansion) for any pair of integers m, n.' (5.1)

For instance, the '1-term' outer expansion is

$$\phi^{(1)} = [\Psi^{(1)}(x,y)], \tag{5.2}$$

which has a '2-term' inner expansion

$$\Psi^{(1)}(x,0) + |y|\,\Psi_y^{(1)}(x,0_+) \tag{5.3}$$

from (3.13), where $\Psi_y^{(1)}(x,0_+)$ is an arbitrary function of x while $\Psi^{(1)}(x,0)$ is determined from $\Psi_y^{(1)}$ by (3.14). On the other hand the '2-term' inner expansion is

$$\Phi^{(1)} + \Phi^{(2)} = [f_1(x)] + [f_2(x) + \Phi_*^{(2)}], \tag{5.4}$$

which has a '1-term' outer expansion

$$f_1(x) + u(x)\,|y| \tag{5.5}$$

from equation (4.12).

Then equating the expressions (5.3) and (5.5) gives

$$\Psi_y^{(1)}(x,0_+) = u(x) = (1/2h)\,S'(x) \tag{5.6}$$

and

$$f_1(x) = \Psi^{(1)}(x,0). \tag{5.7}$$

In aerodynamic terms equation (5.6) may be interpreted as an indication that, as far as the outer expansion is concerned, the ship looks like a symmetrical two-dimensional thin wing of thickness $S(x)/h$, which is physically plausible since it may be observed that this quantity is the mean thickness of the ship averaged over the depth of the channel. For instance, in the case of a vertical strut of thickness $B(x)$ spanning the stream, the Michell solution is recovered exactly since $S = hB$.

Equations (5.7), (5.6) and (3.14) determine $f_1(x)$ completely; specifically we have

$$f_1(x) = \begin{Bmatrix} \dfrac{1}{2\pi h\sqrt{(1-F_h^2)}}\displaystyle\int_{-\infty}^{\infty} d\xi S'(\xi)\log|x-\xi| \quad (F_h < 1) \\ -\dfrac{1}{2h\sqrt{(F_h^2-1)}}S(x) \quad (F_h > 1) \end{Bmatrix}. \tag{5.8}$$

Continuing the matching process, we have the '2-term' outer expansion as

$$\phi^{(1)}+\phi^{(2)} = \Psi'^{(1)}(x,y)+\Psi'^{(2)}(x,y) \tag{5.9}$$

with a '3-term' inner expansion

$$[f_1]+[u|y|+\Psi'^{(2)}(x,0)]+[\,|y|\,\Psi'^{(2)}_y(x,0_+)-\tfrac{1}{2}y^2(1-F_h^2)f_1''] \tag{5.10}$$

(where we have substituted for $\Psi'^{(1)}$ and $\Psi'^{(1)}_y$ the values from (5.6) and (5.7)). On the other hand the '3-term' inner expansion is

$$\Phi^{(1)}+\Phi^{(2)}+\Phi^{(3)} = [f_1(x)]+[f_2(x)+\Phi^{(2)}_*]$$
$$+[f_1'\Phi^{(2)}_*-\tfrac{1}{2}f_1''[(1-F_h^2)y^2+F_h^2(z+h)^2]+f_3(x)], \tag{5.11}$$

with a '2-term' outer expansion,

$$[f_1+u|y|]+[f_2+f_1'u|y|-\tfrac{1}{2}f_1''(1-F_h^2)y^2]. \tag{5.12}$$

The expressions (5.10) and (5.12) must be identical, which will be achieved if

$$\Psi'^{(2)}_y(x,0_+) = uf_1'$$
$$= f_1'(x)\,S'(x)/h, \tag{5.13}$$

and

$$f_2(x) = \Psi'^{(2)}(x,0). \tag{5.14}$$

Thus $f_2(x)$ is completely determinate in principle via equation (3.15) with $y = 0$; however, in practice except when $F_h = 0$, equation (3.15) remains too complicated for explicit computations. If $F_h = 0$ then the relation between $\Psi'^{(2)}$ and $\Psi'^{(2)}_y$ is still (3.14) so that $f_2(x)$ is obtained by replacing S' by $f_1'S'$ in equation (5.8).

6. The inner expansion of the pressure and forces

The hydrodynamic pressure is obtained from Bernoulli's equation

$$-p/(\tfrac{1}{2}\rho U^2) = 2\phi_x+\phi_x^2+\phi_y^2+\phi_z^2. \tag{6.1}$$

Substituting the '3-term' inner expansion (5.11) into (6.1) gives

$$-\frac{p}{\frac{1}{2}\rho U^2} = [2f_1']+[(f_1')^2+2\Phi^{(2)}_{*x}+(\Phi^{(2)}_{*y})^2+(\Phi^{(2)}_{*z})^2+2f_2']+O(\epsilon^3), \tag{6.2}$$

i.e.

$$p = [p_1(x)]+[p_2(x)+P_2(y,z;x)]+O(\epsilon^3), \tag{6.3}$$

where

$$p_1(x) = -\rho U^2 f_1'(x), \tag{6.4}$$

$$p_2(x) = -\rho U^2[f_2'(x)+\tfrac{1}{2}(f_1'(x))^2], \tag{6.5}$$

$$P_2(y,z;x) = -\rho U^2[\Phi^{(2)}_{*x}+\tfrac{1}{2}(\Phi^{(2)}_{*y})^2+\tfrac{1}{2}(\Phi^{(2)}_{*z})^2]. \tag{6.6}$$

Thus the pressure is composed of two parts, both of which are expanded into an asymptotic series in ϵ. The 'interaction' pressure

$$p_1(x)+p_2(x)+\dots \tag{6.7}$$

is a function of x only and measures interactions between sections (at least when $F_h < 1$) since it is defined by integral transforms like (5.8). On the other hand the 'non-interaction' pressure

$$P_2(y,z;x)+P_3(y,z;x)+\dots \tag{6.8}$$

varies around the cross-section, and is calculated by solving a sequence of Neumann problems in each cross-sectional plane separately. There is no interaction between the pressure P_2 at one section x and that at another, except in the sense that the formula (6.6) involves the first derivative $\Phi^{(2)}_{*x}$ of $\Phi^{(2)}_{*}$ with respect to x.

To first order the pressure is dominated by $p_1(x)$, i.e.

$$p = -\rho U^2 f_1'(x) + O(\epsilon^2) \tag{6.9}$$

$$= \begin{cases} \dfrac{-\rho U^2}{2\pi h\sqrt{(1-F_h^2)}}\displaystyle\int_{-\infty}^{\infty} d\xi \frac{S'(\xi)}{x-\xi} & (F_h < 1) \\ \dfrac{\rho U^2}{2h\sqrt{(F_h^2-1)}} S'(x) & (F_h > 1) \end{cases}. \tag{6.10}$$

Physically, this may be explained by noting that (6·9) would follow by neglecting all but the streamwise component of the disturbance velocity. A slender body in a shallow stream causes disturbance velocities of equal orders of magnitude in all directions; under such circumstances the velocity increment or decrement in the *streamwise* direction contributes *more* to the disturbance pressure than transverse disturbances. Shallowness is essential to this argument; if the stream is not shallow the streamwise disturbance of a slender body is an order of magnitude smaller than transverse disturbances, which leads to contributions to the pressure of *equal* orders of magnitude from all 3 disturbance components.

Some further comments on the result (6.10) are appropriate. The fact that the first-order pressure is dependent only on x implies, for instance, that the pressure is predominantly constant around the cross-section of the hull, irrespective of the shape of the section. Any dependence on the shape of the section can arise formally only with the term $P_2(y,z;x)$ of the second approximation, whereas the only information from the hull geometry needed to calculate $p_1(x)$ is the cross-sectional area curve $S(x)$. Further, we may observe that $p_1(x)$ is the hydrodynamic pressure everywhere in the fluid at cross-section x, even on the bottom. One might expect a very large velocity and hence abnormally low pressure at any point where a cross-section almost touches bottom, but the conclusion from the present analysis is that to first order the pressure at such a point is no lower than anywhere else on the same cross-section. Presumably this implies that the fluid passes to the side of any such close gap so as to keep the velocity there comparable with that elsewhere on the cross-section. Of course there is nothing to prevent the general order of the pressure over such a cross-section from being lower than that on a neighbouring cross-section; however, any such effect is not critically dependent on there being only a small distance to the bottom locally, but rather on the overall distribution of cross-sectional area of the hull relative to the depth of water.

Let us now use the expression (6.10) for the first-order pressure on the hull to calculate *forces* to first order. Now it is easy to show (Newman & Tuck 1964) that if $B(x)$ is the beam of the ship at section x (i.e. the intersection of the section at x with the plane $z = 0$), and $S(x)$ is the previously defined cross-sectional area of the section at x, then

$$-\iint p_1(x)\,d\mathbf{S} = \mathbf{i}\int dx p_1(x) S'(x) + \mathbf{k}\int dx p_1(x) B(x) \tag{6.11}$$

for any function $p_1(x)$, the double integral being taken over the portion of the hull below the plane $z = 0$ and the single integrals along the length of the ship. Equation (6.11) is an identity and does not require any slenderness assumption.

However, if the ship is slender, the left-hand side of (6.11) is the first-order hydrodynamic force on the ship. Thus we can assert that to first order the hydrodynamic force on the ship consists of a vertical component $p_1(x)B(x)$ per unit length and a streamwise component $p_1(x)S'(x)$ per unit length.

Thus, to first order the vertical force is

$$F = -\rho U^2 \int dx f_1'(x) B(x) \tag{6.12}$$

(positive upwards), the trim moment about the y-axis is

$$M = \rho U^2 \int dx\, x f_1'(x) B(x) \tag{6.13}$$

(positive with bow up, stern down), and the wave resistance is

$$R = -\rho U^2 \int dx f_1'(x) S'(x). \tag{6.14}$$

Finally, on substituting the expression (5.8) for $f_1(x)$ and further manipulating the resulting double integral, we find in the subcritical case $F_h = U/\sqrt{(gh)} < 1$,

$$F = \frac{\rho U^2}{2\pi h\sqrt{(1-F_h^2)}} \iint dx\, d\xi\, B'(x) S'(\xi) \log|x-\xi|, \tag{6.15}$$

$$M = -\frac{\rho U^2}{2\pi h\sqrt{(1-F_h^2)}} \iint dx\, d\xi (xB(x))' S'(\xi) \log|x-\xi|, \tag{6.16}$$

$$R = 0, \tag{6.17}$$

while in the supercritical case $F_h > 1$,

$$F = \frac{\rho U^2}{2h\sqrt{(F_h^2-1)}} \int dx\, S'(x)\, B(x), \tag{6.18}$$

$$M = -\frac{\rho U^2}{2h\sqrt{(F_h^2-1)}} \int dx\, S'(x)\, xB(x), \tag{6.19}$$

$$R = \frac{\rho U^2}{2h\sqrt{(F_h^2-1)}} \int dx\, (S'(x))^2. \tag{6.20}$$

Equations (6.17) and (6.20) are generalizations of Michell's result (1.1). Thus we find (as we should expect from the aerodynamic analogy) that the subcritical wave resistance is always zero to first order, while the supercritical resistance is a positive definite expression which reduces to that given by Michell for the case of a vertical strut, where

$$S(x) = hB(x).$$

Since we still have a result which, from the point of view of the naval architect, is unreasonable in that the wave resistance is zero in the more important subcritical range, it would appear desirable to pursue the study of the wave resistance to a second approximation. There is every reason to expect a non-zero result at second order, since we shall then be introducing some effects which may be described as finite-depth effects, and which are known to give a non-zero wave resistance. One should observe that, while the aerodynamic analogy no longer applies exactly at second order, the behaviour of the wave resistance of a ship in shallow water will be qualitatively similar to that of an airplane passing through the sound barrier, involving a very sharp rise in resistance as the critical velocity is approached. The phenomenon may be experienced by paddling a canoe in shallow water.

As mentioned earlier, carrying the theory to second order is a formidable if worthwhile task, and we shall not attempt to go further than the first-order results in the present paper. However, whereas the first-order results for wave resistance are rather disappointing, the results for the vertical force and moment are of considerable interest. Instead of investigating the formulae for these quantities in detail, it is somewhat more convenient to use Archimedes principle to give the resulting vertical displacement (sinkage) and trim angle, and this is done in the following section.

7. Sinkage and trim

The vertical force and moment given by equations (6.15)–(6.19) may be written in the form

$$-F = \rho g L \left(\int B(x)\, dx \right) \frac{F_h^2}{\sqrt{(|1-F_h^2|)}} C_F, \tag{7.1}$$

$$M = \rho g \left(\int x^2 B(x)\, dx \right) \frac{F_h^2}{\sqrt{(|1-F_h^2|)}} C_M, \tag{7.2}$$

where C_F, C_M are non-dimensional coefficients dependent only on the geometry of the ship, but taking different values according to whether $F_h \gtrless 1$. For $F_h < 1$ we have

$$C_F = -\frac{1}{2\pi L} \iint dx\, d\xi\, B'(x)\, S'(\xi) \log|x-\xi| \bigg/ \int dx\, B(x), \tag{7.3}$$

$$C_M = -\frac{1}{2\pi} \iint dx\, d\xi (xB(x))'\, S'(\xi) \log|x-\xi| \bigg/ \int dx\, x^2 B(x), \tag{7.4}$$

while for $F_h > 1$

$$C_F = -\frac{1}{2L} \int dx\, B(x)\, S'(x) \bigg/ \int dx\, B(x), \tag{7.5}$$

$$C_M = -\frac{1}{2} \int dx\, xB(x)\, S'(x) \bigg/ \int dx\, x^2 B(x). \tag{7.6}$$

Suppose now the ship responds to these forces, and experiences a 'sinkage' s defined as the downward vertical displacement at $x = 0$, and a 'trim' t defined as the bow-up angle of rotation about $x = 0$. Then to first order in ϵ, Archimedes' principle gives

$$\rho g \int (s+xt)\, B(x)\, dx = -F, \tag{7.7}$$

$$\rho g \int (s+xt)\, xB(x)\, dx = M. \tag{7.8}$$

These are simultaneous equations which may be solved to give s and t. It is again convenient to write the solution in non-dimensional form, setting

$$s/L = C_S F_h^2/\sqrt{|1-F_h^2|}, \tag{7.9}$$

$$t = C_T F_h^2/\sqrt{|1-F_h^2|}, \tag{7.10}$$

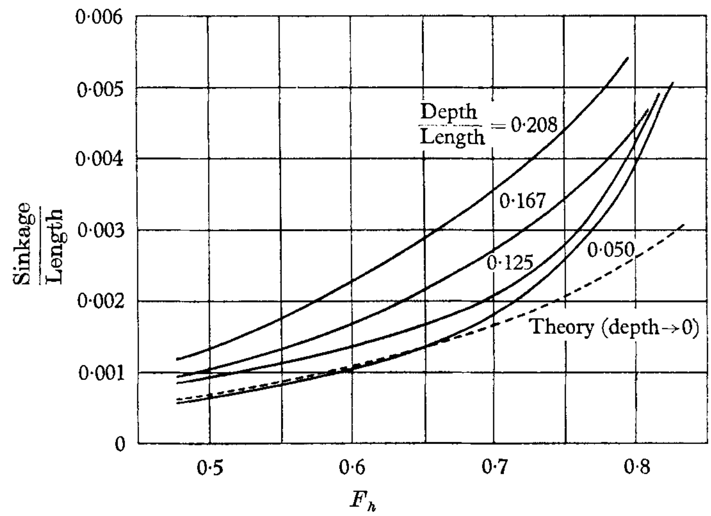

FIGURE 2. Subcritical sinkage according to theory and experiment at various depths.

where the coefficients C_S, C_T are related to the force and moment coefficients C_F, C_M by the equations

$$C_S = (C_F - \alpha C_M)/(1-\alpha\beta), \tag{7.11}$$

$$C_T = (C_M - \beta C_F)/(1-\alpha\beta), \tag{7.12}$$

with

$$\alpha = \int xB(x)\, dx \Big/ L \int B(x)\, dx, \tag{7.13}$$

$$\beta = L \int xB(x)\, dx \Big/ \int x^2 B(x)\, dx. \tag{7.14}$$

A computer program has been developed to evaluate the coefficients C_F, C_M, C_S, C_T for any ship, defined by given functions $S(x)$, $B(x)$.

It may be observed that neither the speed U nor the depth h occurs explicitly in any of equations (7.1), (7.2), (7.9) or (7.10), but only through the combination

$$F_h^2/\sqrt{|1-F_h^2|} = U^2/\sqrt{|U^2-gh|}. \tag{7.15}$$

Thus within either the sub- or supercritical range, the variation with speed of (say) the sinkage is always given by the expression (7.15), irrespective of the shape of the ship or the depth of the water. If we were to plot sinkage against speed U, changing the shape of the ship would change only the vertical scale, while changing the depth would change only the horizontal scale; clearly it is more desirable to plot against Froude number F_h, in which case a unique curve is obtained for a given ship independent of the depth h.

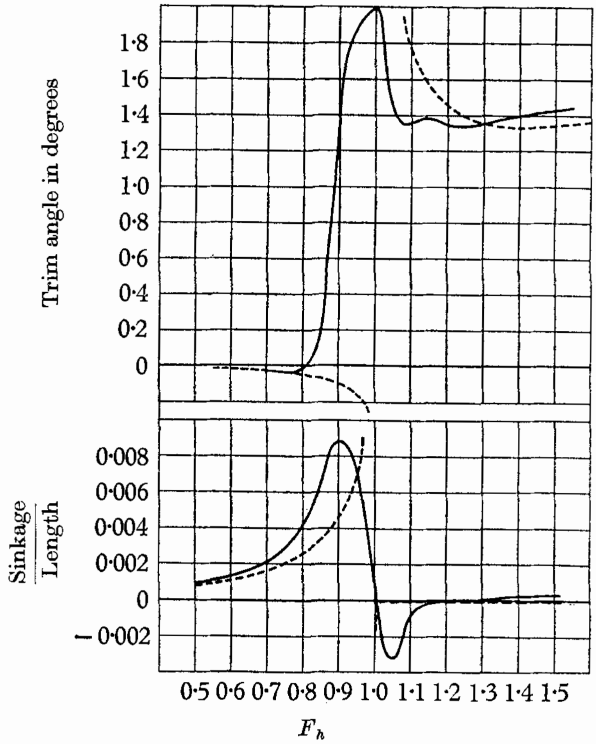

FIGURE 3. Comparison between theoretical and experimental sinkage and trim at $h/L = 0{\cdot}125$. ----, Theory; ——, experiment.

Figure 2 shows such a curve for the theoretical subcritical sinkage, together with some experimental results reported by Graff, Kracht & Weinblum (1964) from tests with a model of the same ship. The experiments were performed at a number of different depths, and curves are shown for values of h/L of 0·05, 0·125, 0·167 and 0·208. The first-order shallow-water theory predicts a unique curve, but this curve is clearly just the limit as $h \to 0$ with F_h held fixed; further terms in the asymptotic expansion would provide an explicit dependence on h. The agreement is quite good at the lowest depth tested ($h/L = 0{\cdot}05$) for $F_h < 0{\cdot}7$, but deteriorates both as the depth increases and as the Froude number gets nearer to unity. Fortunately it is relatively low values of both h/L and F_h which are of practical importance.

It is clear that good agreement cannot be expected near to the critical speed, where the first-order theory predicts infinite values for all forces. By analogy with the aerodynamics of transonic flow, we should expect that in order to predict correctly the finite values obtained in this region we should need to consider some special non-linear effects, and this will not be done here. The actual behaviour found by Graff *et al.* through the critical region is shown in figure 3 for the case $h/L = 0{\cdot}125$. One effect of non-linearity near critical speed seems to be to force the trim to take up its very large supercritical value a little below critical speed; this quickly swamps the small subcritical trim, and at about $F_h = 0{\cdot}9$ the supercritical value is already achieved. At the same Froude number the sinkage has reached its maximum, and is about to decrease as non-linear 'trans-critical' effects push it towards its low supercritical value.

Figure 3 also shows an interesting qualitative feature of measured sinkage and trim values, which is confirmed by the theoretical curves given, namely, the fact that sinkage is the dominant phenomenon at subcritical speeds, whereas trim is dominant at supercritical speeds. Further, it is found by experiment that the large subcritical sinkage is always positive (i.e. downward) and the supercritical trim is likewise positive (i.e. bow-up). The theory predicts this behaviour explicitly in two special cases. First, if the ship has fore-and-aft symmetry it is possible to prove that

$$C_T = C_M = 0 \quad \text{for} \quad F_h < 1,$$

and

$$C_S = C_F = 0 \quad \text{for} \quad F_h > 1,$$

so that for such a symmetrical ship we predict a zero subcritical trim and zero supercritical sinkage. Secondly, there exists a non-trivial class of ships ('simple ships') such that $S(x)/B(x) = \text{const.}$ For such ships we can prove rigorously that

$$C_F > 0 \quad \text{for} \quad F_h < 1,$$

and

$$C_M > 0 \quad \text{for} \quad F_h > 1,$$

so that simple ships must experience a downward force in the subcritical range and a bow-up moment in the supercritical range. Although practical ship shapes are neither symmetrical nor 'simple', they are sufficiently close to being so, that the above results retain a qualitative validity in the general case.

REFERENCES

GRAFF, W., KRACHT, A. & WEINBLUM, G. 1964 Some extensions of D. W. Taylor's standard series. *Trans. Soc. Nav. Arch. Mar. Engrs.*, **72**, 374.

KOSTYUKOV, A. A. 1959 *Theory of Ship Waves and Wave Resistance*. Leningrad.

MICHELL, J. H. 1898 The wave-resistance of a ship. *Phil. Mag.* **45**, 106.

NEWMAN, J. N. & TUCK, E. O. 1964 Current progress in the slender body theory for ship motions. *Proc. Fifth Symposium on Naval Hydrodynamics*, Bergen, Norway, September 1964.

TUCK, E. O. 1964 A systematic asymptotic expansion procedure for slender ships. *J. Ship Res.* **8**, no. 1, 15.

VAN DYKE, M. D. 1964 *Perturbation Methods in Fluid Mechanics*. New York: Academic Press.

WEHAUSEN, J. V. & LAITONE, E. V. 1960 *Handb. Phys.*, vol. 9, *Surface Waves*. Berlin: Springer-Verlag.